Stability of cosmological detonation fronts

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Abstract

We present results of a linear stability analysis of relativistic detonation fronts, which have been considered as representing phase interfaces in cosmological first order phase transitions. After discussing general stability conditions for detonation fronts, we concentrate on the properties of the fronts with respect to corrugation instabilities and discuss separately the cases of Chapman-Jouguet and strong detonation waves. Contrarily to what recently claimed, we find that strong detonations are both evolutionary and stable with respect to corrugations of the front. Moreover, Chapman-Jouguet detonations appear to be unconditionally linearly stable. The implications of the stability results for first order cosmological phase transitions are presented and a discussion of the causal structure of reaction fronts is also given.

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I. Introduction

Reaction fronts within cosmological scenarios have been the subject of extensive recent investigation. In the classical treatment they are described as moving surfaces by means of which a suitable gas mixture undergoes a chemical transformation with liberation of heat and with the gas being either accelerated and decompressed or decelerated and compressed as it passes through the front. The study of the microphysics in the narrow region where the reaction processes take place is extremely complicated and an exhaustive theory of it within the present context has not been reached yet. Nevertheless, a satisfactory hydrodynamical description of reaction fronts can be achieved when these are treated as discontinuity surfaces of infinitesimal and constant width across which rapid changes in the fluid variables occur. This approximation is certainly good if the front has a thickness which is much smaller than the typical length scale for the variation of the flow variables and if the thermal and the viscous time scales are much smaller than the one set by the motion of the front (we shall assume that these requirements are met within the scenarios of cosmological phase transitions considered here). In this respect, reaction fronts are very similar to the better known shock fronts (with which they share some properties) and can be described by means of the same mathematical theory [1, 2, 3, 4, 5].

Reaction fronts can occur in nature in connection with a variety of different classes of phenomena and in general they can be distinguished into *deflagrations* and *detonations* according to whether the front is subsonic or supersonic relative to the medium ahead of the front.[1] Deflagrations (and detonations) can be further classified as *weak* or *strong* according to whether the velocity of the medium behind is subsonic (supersonic) or supersonic (subsonic)[2]. As a general rule, the fluid velocity entering a deflagration front is always smaller than the fluid velocity going out of it, while the opposite is true for a detonation front. An additional and special class of reaction fronts is the one for which the velocity of the fluid out of the front is

[1]The flow regions ahead of and behind a propagating front will be here also referred to as the "upstream" and the "downstream" regions respectively.

[2]Unless specified, all the velocities are meant to be referred to the front rest frame.

exactly equal to the local sound speed. These fronts are called *Chapman-Jouguet* deflagrations/detonations, and represent a specifically interesting class of phenomena. (The classification of the various reaction fronts is summarized in Table I, where v_1 is the fluid velocity ahead of the front, v_2 the fluid velocity behind the front and c_{s1}, c_{s2} are the sound speeds on either side).

It can be shown that Chapman-Jouguet processes yield stationary values both for the front velocity relative to the fluid ahead and for entropy of the fluid which has gone across the front [1]. More precisely, v_1 and the entropy of the fluid behind the front are at a maximum in the case of a *Chapman-Jouguet deflagration*, while v_1 and the entropy of the fluid behind are at a minimum in the case of a *Chapman-Jouguet detonation*, which is then the slowest of all possible detonations. Chapman-Jouguet detonations represent a particularly relevant class of reaction fronts, for which the speed of the front is completely determined in terms of the boundary conditions and of the energy–momentum conservation. This privileged nature is furthermore underlined in the "Chapman-Jouguet hypothesis", according to which detonations in chemical burning should occur *only* under the form of Chapman-Jouguet detonations. A proof of this can be found in [1, 2] but it is important to stress that because of the differences between chemical combustion and phase transitions, the validity of the Chapman-Jouguet hypothesis cannot be extended to the context of cosmological phase transitions [6].

Although in nature detonations in chemical burning often appear accompanied by nonlinear effects (such as transverse shock waves or turbulence [4]), it seems that the hypothesis is generally verified to a good approximation, with detonation fronts which although complicated, propagate at a constant velocity close to the theoretical Chapman-Jouguet value [7, 8, 9].

Because of their properties, reaction fronts have been considered for studying the hydrodynamics of the phase interface during cosmological first order phase transitions [10] such as the electroweak [11] and the quark-hadron transition (see [6] for detonations and [12] for a review of deflagrations). In general, first order cosmological phase transitions start with the nucleation of bubbles of the low temperature

	DEFLAGRATIONS ($v_1 < v_2$)	DETONATIONS ($v_1 > v_2$)
Weak	$v_1 < c_{s1}, \;\; v_2 < c_{s2}$	$v_1 > c_{s1}, \;\; v_2 > c_{s2}$
Chapman–Jouguet	$v_1 < c_{s1}, \;\; v_2 = c_{s2}$	$v_1 > c_{s1}, \;\; v_2 = c_{s2}$
Strong	$v_1 < c_{s1}, \;\; v_2 > c_{s2}$	$v_1 > c_{s1}, \;\; v_2 < c_{s2}$

Table I. The various combinations of the fluid velocities for the different types of reaction front. The velocities are referred to the front rest frame, with v_1 being the fluid velocity ahead of the front and v_2 the fluid velocity behind the front. Similarly c_{s1} and c_{s2} are the sound speeds on either side of the front.

phase within a supercooled ambient medium. The surface that separates the two coexisting phases is then induced into motion (the new phase is thermodynamically favoured) and in doing this it transforms one phase into the other. An aspect which requires great attention when studying the evolution of *thermodynamically stable* reaction fronts is that of *hydrodynamic stability* and this represents a large area of research both from the experimental and the theoretical point of view.

The stability of classical deflagration fronts was first studied by Landau in a seminal work of 1944 [13]. Following this, a number of special relativistic linear stability analyses have been performed by several authors both in the limit of small velocities [14] and in the case of small and large velocities [15]. These works had a direct counterpart in the numerous studies produced on the hydrodynamics of a cosmological first order phase transition in which the phase interface moves as a weak deflagration front (see [16, 17, 18, 19] for the quark-hadron transition).

The results of these stability analyses have shown that, within a linear theory, cosmological hadron bubbles could be unstable on time scales much smaller than the typical time scale discussed for the duration of the phase transition (the situation is different for the electroweak transition, where the bubble wall seems stable under hydrodynamic perturbations [15]). It is well known however, that the ultimate onset

of the instabilities cannot be fully assessed within a linear analysis, since it might also be that the instability modes are controlled by intervening nonlinear effects which would limit the energy transfer into the unstable modes. Of course, a necessary and sufficient condition for the validity of the above arguments is that a mutual causal connection is maintained between the front and the upstream or the downstream regions of the flow.

The stability of classical detonation fronts has been studied quite extensively in the one-dimensional and linear regime [20] and attempts are currently being made to extend this analysis within a weakly nonlinear theory (see [21] for a list of references). A recent investigation of the stability properties of relativistic detonation fronts in cosmological phase transitions has been made by Abney [22] who has concentrated particularly on the case of Chapman-Jouguet detonations. As pointed out in [22], while instability modes are not allowed in the upstream flow region of a detonation front, it is not possible to exclude them in the region behind the front, where they could also grow exponentially with time. The present work aims to reconsider in more detail the analysis performed in [22] both in the specific case of Chapman-Jouguet detonations and in the general case of strong detonations.

In order to do this, we start in Section II by presenting the general equations deduced from the standard linear stability analysis of special relativistic flows. In Section III we examine the stability of generic discontinuity fronts with respect to corrugations and discuss the application of these results to both Chapman-Jouguet and strong detonations in Section IV. Section V is dedicated to the analysis of the boundary conditions that detonation fronts need to satisfy in a cosmological first order phase transition and there we comment on the implications of the causal structure of weak detonations for their stability properties. Conclusions are finally presented in Section VI. We here adopt units for which $c = 1$, greek indices are taken to run from 0 to 3 , while latin indices from 1 to 3. The metric has signature $(-, +, +, +)$ and commas in covariant notation are used to denote standard partial derivatives.

II. Linear hydrodynamic stability

We here discuss the linear stability analysis of a relativistic planar detonation wave. The procedure followed for the derivation of the set of perturbed hydrodynamical equations parallels in part that presented in [22] (where Chapman-Jouguet detonations only were considered), but some important differences will emerge in the course of the discussion.

Consider a plane detonation front which is propagating in a Minkowski space-time with $(t,\, x,\, y,\, z)$ being the inertial coordinates. The hydrodynamics of an unperturbed detonation wave can be described as a one dimensional flow in which the whole space-time is divided in two half spaces separated by a discontinuity surface moving at four-velocity $u^\mu_s = \gamma_s(1,\, v^i_s)$, with $\gamma_s = (1 - v^i_s\, v^j_s\, \delta_{ij})^{-1/2}$ and $v^i_s = dx^i_s/dt$.

In this case, it is always possible to perform a Lorentz transformation by means of which the front at any instant is taken to be at rest on the $(y,\, z)$ plane and there are no three-velocity components tangent to the front (hereafter we shall refer to the three-velocity vectors simply as velocities). In this comoving reference frame, we can think of the front as a planar discontinuity surface which divides the three-space into the upstream region 1 and downstream region 2 and which is crossed by positive velocities from left to right when the front is left propagating in the inertial frame (see Figure 1).

Across this surface matter undergoes either a combustion or a phase transformation (*e.g.* passing from unconfined quarks to light hadrons in the case of the quark-hadron phase transition). Fluids on either side of the front are assumed to be ideal and described by the standard stress-energy tensor of a relativistic perfect gas

$$T^{\mu\nu} = (e+p)u^\mu u^\nu + pg^{\mu\nu}\ , \tag{1}$$

where $u^\mu \equiv \gamma(1\ ,\vec{v})$ is the fluid four-velocity, e is the energy density, p is the pressure and $g^{\mu\nu}$ is the metric tensor. The hydrodynamical equations can easily be derived from the projection of the four-divergence of the stress-energy tensor along

the direction defined by the fluid four-velocity and orthogonal to it, so as to express local conservation of energy and momentum respectively as

$$u_\mu T^{\mu\nu}{}_{,\,\nu} = 0 \ , \tag{2}$$

$$P_{\alpha\mu} T^{\mu\nu}{}_{,\,\nu} = 0 \ , \tag{3}$$

where $P_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ is the projection operator orthogonal to u^μ (*i.e.* $P_{\mu\nu}u^\nu = 0$). Making use of (1), equations (2) and (3) can be written explicitly as

$$c_s^2 w u^\mu{}_{,\,\mu} + p_{,\,\mu} u^\mu = 0 \ , \tag{4}$$

and

$$w u_{\alpha,\,\mu} u^\mu + u_\alpha p_{,\,\mu} u^\mu + p_{,\,\alpha} = 0 \ , \tag{5}$$

where $w = (e+p)$ is the enthalpy density and $c_s = (\partial p/\partial e)_s^{1/2}$ is the local sound speed (here s is the specific entropy). Equations (4) and (5) represent the usual "zeroth order" hydrodynamical equations and describe the motion of fluid elements on either side of the detonation front. Following standard linear perturbation analysis, we now introduce a small perturbation in the relevant hydrodynamical variables so that, at first order in the expansion, the new perturbed (primed) variables are

$$p \quad \longrightarrow \quad p' = p + \delta p \ , \tag{6}$$

$$u^\mu \equiv \gamma(1,\ \vec{v}) = \gamma(1,\ v_x,\ 0,\ 0) \quad \longrightarrow \quad (u^\mu)' = u^\mu + \delta u^\mu \ , \tag{7}$$

where $\gamma = (1 - v_x^2)^{-1/2}$ and (the flow is taken to be uniform and unperturbed along the z-axis direction),

$$
(u^\mu)' \equiv \gamma(1+\gamma^2 v_x \delta v_x,\ v_x + \gamma^2 \delta v_x,\ \delta v_y,\ 0)\ . \tag{8}
$$

As a result, the perturbed expressions of equations (4) and (5) are

$$
c_s^2 w \left(\gamma^2 v_x \frac{\partial}{\partial t}\delta v_x + \gamma^2 \frac{\partial}{\partial x}\delta v_x + \frac{\partial}{\partial y}\delta v_y\right) + \frac{\partial}{\partial t}\delta p + v_x \frac{\partial}{\partial x}\delta p = 0\ , \tag{9}
$$

$$
\gamma^2 w \left(\frac{\partial}{\partial t}\delta v_x + v_x \frac{\partial}{\partial x}\delta v_x\right) + v_x \frac{\partial}{\partial t}\delta p + \frac{\partial}{\partial x}\delta p = 0\ , \tag{10}
$$

$$
\gamma^2 w \left(\frac{\partial}{\partial t}\delta v_y + v_x \frac{\partial}{\partial x}\delta v_y\right) + \frac{\partial}{\partial y}\delta p = 0\ , \tag{11}
$$

which can also be written, in a more compact form, as

$$
\left(\mathbf{C}_t \frac{\partial}{\partial t} + \mathbf{C}_x \frac{\partial}{\partial x} + \mathbf{C}_y \frac{\partial}{\partial y}\right)\vec{\mathbf{U}} = 0\ , \tag{12}
$$

where

$$
\mathbf{C}_t = \begin{pmatrix} 1 & \gamma^2 w c_s^2 v_x & 0 \\ v_x & \gamma^2 w & 0 \\ 0 & 0 & \gamma^2 w \end{pmatrix}\ , \tag{13}
$$

$$
\mathbf{C}_x = \begin{pmatrix} v_x & \gamma^2 w c_s^2 & 0 \\ 1 & \gamma^2 w v_x & 0 \\ 0 & 0 & \gamma^2 w v_x \end{pmatrix}\ , \tag{14}
$$

$$
\mathbf{C}_y = \begin{pmatrix} 0 & 0 & w c_s^2 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\ , \tag{15}
$$

and $\vec{\mathbf{U}}$ is the state-vector for the perturbations

$$\vec{\mathbf{U}} = \begin{pmatrix} \delta p \\ \\ \delta v_x \\ \\ \delta v_y \end{pmatrix} . \tag{16}$$

The most general solution of (12) has the form

$$\vec{\mathbf{U}}(t,\ x,\ y) = \vec{\mathbf{A}}(x) e^{-i(\omega t + k y)} , \tag{17}$$

with ω being a complex number, k a real number and

$$\vec{\mathbf{A}}(x) = \sum_{j=1}^{3} (c_j \vec{\mathbf{L}}_j) e^{-i(l_j x)} . \tag{18}$$

The l_j are the complex eigenvalues of the secular equation (12), $\vec{\mathbf{L}}_j$ are the corresponding eigenvectors and c_j are three real constant coefficients. Substituting the trial solution (17) in (12), leads to an homogeneous system of equations whose coefficients are in the secular matrix

$$\mathbf{D} \equiv (\mathbf{C}_t\ \omega + \mathbf{C}_x\ l + \mathbf{C}_y\ k) , \tag{19}$$

The eigenvalues l_j can then be found by setting to zero the determinant of $\mathbf{D}$. Doing this we obtain the dispersion relation

$$\det(\mathbf{D}) = (l v_x + \omega) \left[(l v_x + \omega)^2 - (\omega v_x + l)^2 c_s^2 - (1 - v_x^2) k^2 c_s^2 \right] = 0 , \tag{20}$$

which has the roots

$$l_1 = -\frac{\omega}{v_x} , \tag{21}$$

$$l_{2,3} = \frac{1}{v_x^2 - c_s^2} \left\{ (c_s^2 - 1)\omega v_x \pm c_s(1 - v_x^2) \left[\omega^2 + \frac{(v_x^2 - c_s^2)}{1 - v_x^2} k^2 \right]^{1/2} \right\} . \tag{22}$$

A first point to notice is that the eigenvalues $l_{2,3}$ become divergent for a fluid velocity normal to the front (v_x) equal to the local sound speed. The presence of a singularity at the sonic point is a general feature of the linear stability analysis of shock waves and reaction fronts [23, 2] and so it is necessary to take particular care when examining these limiting cases. It is easy to realize however, that the singularity in (22) is only an apparent one and it can be avoided if one solves equation (20) directly with $v_x = c_s$. In this case there are only two roots and the new eigenvalues (which we denote with a bar) then are: $\bar{l}_1 = l_1$ and

$$\bar{l}_2 = \frac{c_s k^2}{2\omega} - \frac{(1 + c_s^2)}{2c_s}\omega . \tag{23}$$

Using the eigenvalue (23) in the downstream region of the flow solves the problem of the singularity at the sonic point, but necessarily restricts the analysis to Chapman-Jouguet detonations only. In order to investigate the stability properties of generic detonation fronts, it is necessary to make use of the solutions (21) and (22) for fluid velocities near to the sound speed. For this purpose we can write the velocity normal to the detonation front as

$$v_x = c_s + \epsilon , \tag{24}$$

with ϵ being a small positive or negative real number. The value of ϵ must in principle lie in the range $0 < \epsilon < (1 - c_s)$ for the upstream region ($x < 0$) and in the range $-c_s < \epsilon < (1 - c_s)$ for the downstream region ($x > 0$). Making use of (24), it is possible to expand both the numerators and the denominators of solutions (21) and (22) around the sonic point so as to obtain the new eigenvalues (marked with a tilde)

$$\tilde{l}_1 \cong -\frac{\omega}{c_s + \epsilon} \ , \tag{25}$$

$$\tilde{l}_2 \cong -\frac{1}{\epsilon(2c_s+\epsilon)} \left\{ \omega \left(1 + c_s^2 - \frac{k^2 c_s^2}{\omega^2} \right) \epsilon + \omega c_s \left[1 - \frac{k^2}{2\omega^2} + \frac{k^4 c_s^2}{2\omega^4(1-c_s^2)} \right] \epsilon^2 \right\} \ , \tag{26}$$

$$\tilde{l}_3 \cong \frac{1}{\epsilon(2c_s+\epsilon)} \left\{ 2\omega c_s (c_s^2 - 1) + \omega \left(3c_s^2 - 1 - \frac{k^2 c_s^2}{\omega^2} \right) \epsilon \right.$$
$$\left. + \omega c_s \left[1 - \frac{k^2}{2\omega^2} + \frac{k^4 c_s^2}{2\omega^4(1-c_s^2)} \right] \epsilon^2 \right\} . \tag{27}$$

Although approximate, these expressions are suitable for a generic value of the fluid velocity near the interface and provide a starting point for the stability analysis of both strong and weak detonations. It is important to notice that $\tilde{l}_2$ is not singular for $\epsilon \to 0$ and that it reduces to $\bar{l}_2$ at first order; for this reason a second order expansion is necessary. This is not the case for $\tilde{l}_3$ which represents the singular root of (22) and for which the first order expansion is, in fact, sufficient.

Next, it is necessary to find the form of the eigenvectors $\vec{\mathbf{L}}_j$ contained in (18). This requires solving the matrix equation

$$\begin{pmatrix} (\omega + \tilde{l}_j v_x) & \gamma^2 w(\omega v_x + \tilde{l}_j) c_s^2 & w c_s^2 k \\ (\omega v_x + \tilde{l}_j) & \gamma^2 w(\omega + \tilde{l}_j v_x) & 0 \\ k & 0 & \gamma^2 w(\omega + \tilde{l}_j v_x) \end{pmatrix} \begin{pmatrix} \vec{\mathbf{L}}_{j\,1} \\ \vec{\mathbf{L}}_{j\,2} \\ \vec{\mathbf{L}}_{j\,3} \end{pmatrix} = 0 \ ,$$
$$j = 1,\ 2,\ 3 \ , \tag{28}$$

which leads to the following eigenvectors

$$\vec{\mathbf{L}}_1 = \begin{pmatrix} 0 \\ \\ 1 \\ \\ -\dfrac{\tilde{l}_1}{k} \end{pmatrix} , \tag{29}$$

$$\vec{\mathbf{L}}_n = \begin{pmatrix} 1 \\ \\ -\dfrac{(\omega v_x + \tilde{l}_n)}{\gamma^2 w(\omega + \tilde{l}_n v_x)} \\ \\ -\dfrac{k}{\gamma^2 w(\omega + \tilde{l}_n v_x)} \end{pmatrix} , \tag{30}$$

$$n = 2,\ 3$$

It is now necessary to ascertain the values of the coefficients c_j for which the solution (17), with eigenvalues (25)–(27) and eigenvectors (29)–(30), satisfies the necessary boundary conditions. For this reason we have to check that, if there are time growing instabilities, the effects of these should be limited in space and not extend to infinity, so that

$$\lim_{x \to \pm\infty} |\vec{\mathbf{U}}(t, x, y)| = 0 \ . \tag{31}$$

After a few algebraic transformations, it is possible to verify that

$$\begin{array}{llcll} \text{(a)} & \mathrm{Im}\ (\omega) > 0 & \Longleftrightarrow & \mathrm{Im}\ (\tilde{l}_1) < 0 \ , & \\ \text{(b)} & \mathrm{Im}\ (\omega) > 0 & \Longleftrightarrow & \mathrm{Im}\ (\tilde{l}_2) < 0 \ , & \\ \text{(c)} & \mathrm{Im}\ (\omega) > 0 & \Longleftrightarrow & \mathrm{Im}\ (\tilde{l}_3) < 0 & \text{for } 0 < \epsilon < (1 - c_s) \ \ , \end{array} \tag{32}$$

and

$$
\begin{array}{llcll}
\text{(d)} & \text{Im }(\omega) > 0 & \Longrightarrow & \text{Im }(\tilde{l}_1) < 0 \ , & \\
\text{(e)} & \text{Im }(\omega) > 0 & \Longrightarrow & \text{Im }(\tilde{l}_2) < 0 \ , & \\
\text{(f)} & \text{Im }(\omega) > 0 & \Longrightarrow & \text{Im }(\tilde{l}_3) > 0 & \text{for } -c_s < \epsilon \leq 0 \ ,
\end{array}
\tag{33}
$$

In order for (31) to be satisfied, it is necessary that Im $(\tilde{l}_j) > 0$ or that the corresponding coefficients c_j are zero for $x < 0$ and that Im $(\tilde{l}_j) < 0$ or $c_j = 0$ for $x > 0$. For modes with Im $(\omega) > 0$ we then have

$$
\begin{array}{lllll}
\text{(a)} & c_1 = c_2 = c_3 = 0 & \text{for } x < 0 \ , & & \\
\text{(b)} & c_1 \neq 0, \ \ c_2 \neq 0, \ \ c_3 = 0 & \text{for } x > 0 \ \text{ and } \epsilon \leq 0 & & (\text{strong}, \ \mathrm{C-J}) \ , \\
\text{(c)} & c_1 \neq 0, \ \ c_2 \neq 0, \ \ c_3 \neq 0 & \text{for } x > 0 \ \text{ and } \epsilon > 0 & & (\text{weak}) \ .
\end{array}
\tag{34}
$$

In other words, the conditions (34) signify that no perturbations can grow ahead of the detonation front (*i.e.* $\vec{\mathbf{U}}(t, x, y) = 0$ for $x < 0$), while this is not necessarily the case for the positive x half-plane, where growing modes are allowed to exist since only one coefficient needs to be zero in the case of strong and Chapman-Jouguet detonations, and none in the case of weak detonations. This latter result represents an important difference between strong and weak detonations and will be further underlined in the next Sections. The condition (34-a) on the coefficients c_j has its physical interpretation in the fact that in the negative x half-plane the flow is supersonic and "entering" the front and, as a consequence, no sonic signal (and therefore no perturbation) can be transmitted upstream of this flow. In the next Section we shall study whether a dynamical evolution of the instabilities behind the detonation front is possible when this is subject to a corrugation instability.

III. Front corrugation stability

Within the theory of shock front stability, it is known that the conditions for a shock to be evolutionary (*i.e.* $v_1 > c_{s1}$, $v_2 < c_{s2}$) [2] are only necessary but not sufficient to prove that it will not develop instabilities. This means that an evolutionary shock (*i.e.* one for which any infinitesimal perturbation of the initial state produces only an infinitesimal change in the flow over a sufficiently short time interval), could become unstable (over a longer time interval) with respect to small perturbations of the discontinuity surface, which would then appear as "corrugations" of the front[3] (see Figure 1).

Corrugation stability analyses of the type presented in this paper have been performed in the past both for a non-relativistic shock [25, 2] and for a relativistic Chapman-Jouguet detonation [22]. A rather different approach for a relativistic shock wave has been proposed by Anile and Russo [26, 23] where the intuitive definition of corrugation stability introduced by Whitham [27] has been translated into a more rigorous form (we recall that according to Whitham, a corrugated shock front is stable if the shock velocity decreases where the front in expanding and increases where it is contracting).

We here discuss the corrugation stability of strong relativistic detonation fronts and show how these relate to the special case of Chapman-Jouguet detonations. The first step consists in establishing the correct eigenvalues to choose. Making use of the conditions (32)–(33), together with (34), it is evident that it is necessary to use the eigenvalues $\tilde{l}_1,\ \tilde{l}_2$ (and the corresponding eigenvectors) in the case of strong and Chapman-Jouguet detonations ($\epsilon \leq 0$) while all of the eigenvalues $\tilde{l}_1,\ \tilde{l}_2,\ \tilde{l}_3$ need to be used in the case of weak detonations ($\epsilon > 0$). The second step consists of requiring that the perturbed hydrodynamical equations satisfy junction conditions at the phase interface expressing the conservation of energy and momentum respectively.

[3]We note that it is possible, in principle, to write necessary and sufficient conditions for a shock wave not to decay into a number of different discontinuity surfaces [24]. However, these conditions do not provide information about the evolution of the shock when this is subject to corrugations of the front.

In a Minkowski spacetime these reduce to (see [28, 29, 30] for a general relativistic treatment)

$$\left[\gamma^2 w v_x\right]_{1,2} = 0 , \tag{35}$$

$$\left[\gamma^2 w v_x^2 + p\right]_{1,2} = 0 , \tag{36}$$

$$\left[v_y\right]_{1,2} = 0 , \tag{37}$$

where $[A]_{1,2} = A_1 - A_2$ (note that these junction conditions need always to be expressed in the front rest frame and that the latter coincides with the coordinate frame only when the front is unperturbed). We next perturb the front with a periodic oscillation in the y-axis direction of the type (see Figure 1)

$$\Delta = \Delta_0 e^{-i(\omega t + ky)} , \tag{38}$$

and calculate the resulting form of the perturbed junction conditions. For this purpose it is convenient to introduce orthogonal unit three-vectors normal ($\underline{\mathbf{n}}$) and tangent ($\underline{\mathbf{t}}$) to the front (Figure 1) which, at the first order in the perturbation, have components

$$\underline{\mathbf{n}} \equiv (1,\ -\frac{\partial \Delta}{\partial y},\ 0) = (1,\ ik\Delta,\ 0) , \qquad \underline{\mathbf{t}} \equiv (\frac{\partial \Delta}{\partial y},\ 1,\ 0) = (-ik\Delta,\ 1,\ 0) . \tag{39}$$

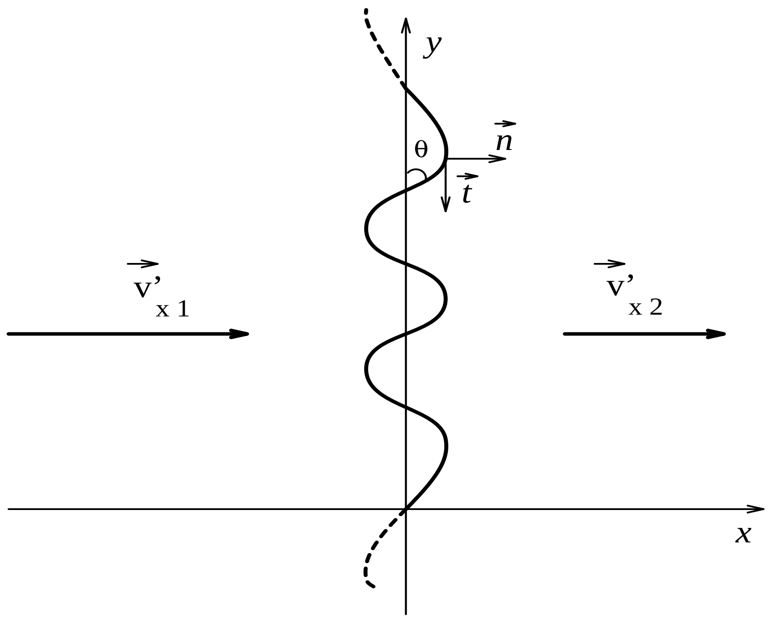


Figure 1. Schematic representation of a corrugated detonation front. $\underline{\mathbf{n}}$ and $\underline{\mathbf{t}}$ are orthogonal unit three-vectors, normal and tangent respectively to the discontinuity surface.

It is now necessary to evaluate the perturbed expressions for the fluid velocities on either side of the front, as these are viewed in the front rest frame. For this purpose it is necessary to perform a relativistic velocity transformation with respect to the detonation velocity $\vec{v}_f = (\partial\Delta/\partial t)\,\underline{\mathbf{n}} = (-i\omega\Delta)\,\underline{\mathbf{n}}$, so as to obtain the following expressions for the perturbed normal and tangential velocities relative to the front

$$\vec{v}_j{}' \cdot\ \underline{\mathbf{n}} = \left(\frac{v_{xj} + \delta v_{xj} + i\omega\Delta}{1 + i\omega v_{xj}\Delta},\ \delta v_{yj},\ 0\right) \cdot\ \underline{\mathbf{n}} \cong v_{xj} + \delta v_{xj} + \frac{i\omega\Delta}{\gamma_j^2}\ , \tag{40}$$

$$\vec{v}_j{}' \cdot\ \underline{\mathbf{t}} \cong \delta v_{yj} - ikv_{xj}\Delta\ , \tag{41}$$

$$j = 1,\ 2\ \ ,$$

and to which corresponds a perturbed gamma factor

$$\delta\gamma_j \cong \gamma_j^3 v_{xj} \left(\delta v_{xj} + \frac{i\omega\Delta}{\gamma_j^2} \right) , \qquad j = 1,\ 2 \ . \tag{42}$$

It is now convenient to introduce the new state-vector of the perturbations near to the interface and in downstream region of the flow

$$\vec{\mathbf{V}}_2(t,\ y) = \begin{pmatrix} \delta p_2 \\ \\ \delta v_{x2} \\ \\ \delta v_{y2} \end{pmatrix} , \tag{43}$$

where, as discussed in the previous Section, each component of the corresponding $\vec{\mathbf{V}}_1(t,\ y)$ is automatically zero. Making use of (40)–(42) in (35)–(37), it is possible to write out the perturbed junction conditions and to use the resulting system of three equations in order to derive the components of $\vec{\mathbf{V}}_2(t,\ y)$. In doing this, we should also modify the momentum balance across the interface by taking into account the response of the front which is due to its surface tension σ (the energy balance at the front is unaffected by a constant surface tension [31, 28]). This contribution appears in the expressions for the negative x half-plane with the form $\sigma(\partial^2/\partial y^2 - \partial^2/\partial t^2)\Delta$, where the first term is related to the surface curvature, while the second is related to its "inertia". Omitting here the lengthy algebra, the three components of $\vec{\mathbf{V}}_2(t,\ y)$ are found to be

$$\delta p_2 = \frac{1+v_2^2}{\Gamma_- + \Gamma_+ v_2^2} \left[2i\omega w_2 \frac{(v_1 - v_2)(1 - v_1 v_2)}{(1+v_2^2)} \frac{\gamma_2^2 v_2}{v_1} + \sigma(\omega^2 - k^2) \right] \Delta \ , \tag{44}$$

$$\delta v_{x2} = -\frac{1-v_2^2}{\Gamma_- + \Gamma_+ v_2^2} \left[i\omega \frac{(v_1 - v_2)(1 - v_1 v_2)}{v_1} \Gamma_+ + \sigma(\omega^2 - k^2) \frac{\Theta_2}{w_2} v_2 \right] \Delta \ , \tag{45}$$

$$\delta v_{y2} = -ik\Delta(v_1 - v_2)\ , \tag{46}$$

where $\Gamma_\pm = (1 \pm \Theta_2 \gamma_2^2 v_2^2)$, with $\Theta_2 = (\delta w_2/\delta p_2) = 1 + 1/c_{s_2}^2$ for a relativistic gas (note that for compactness we will write $v_{1,2} \equiv v_{x1,x2}$ for the zeroth order velocities). Expressions (44)–(46) represent the special relativistic generalization of the equivalent expressions discussed in §90 of [2] and reduce to these when the Newtonian limit is taken. It is important to remark that the term $(\Gamma_- + \Gamma_+ v_2^2) = 1 + v_2^2(1 - \Theta_2)$, in the denominators of (44)–(45), vanishes whenever $v_2 = c_{s2}$, giving a singular behaviour at the sonic point similar to the one seen in the Newtonian case [2]. We note that the corresponding expressions derived in [22], differ from (44)–(45) and do not show this singular behaviour except in their Newtonian limit. This discrepancy is due to the fact that in Abney's treatment the transformation to the front rest frame is a simple Galileian one and that, also, the perturbation in the squared gamma factor is neglected. Unfortunately, we believe that these omissions, which radically change the nature of the solution at the sonic point and strongly influence the analysis, cannot be considered satisfactory.

As mentioned in Section II, a singular behaviour for a velocity behind the front equal to the local sound speed is a standard feature of the stability analysis of discontinuity surfaces. Nevertheless, great care must be taken when discussing these limiting cases, such as the present Chapman-Jouguet detonations. Some physical insight into the properties of Chapman-Jouguet detonations can already be gained when looking at the perturbed expressions for the hydrodynamical variables in terms of the perturbation Δ [*i.e.* inverting expressions (44)–(45)].

In this case, in fact, we could conclude that the corrugations produced on a Chapman-Jouguet detonation front by perturbations in the downstream fluid variables, are always zero and independent of the strength of the perturbations (*i.e.* independent of the magnitude of $\vec{\mathbf{V}}_2$). In other words, expressions (44)–(45) seem to indicate that at linear order a Chapman-Jouguet detonation is *unconditionally stable* (an identical conclusion will be drawn also from the study of the dispersion relation for a Chapman-Jouguet detonation in Section IV).

At this stage it is possible to deduce the form of the dispersion relation by requiring that the hydrodynamical perturbations present in the fluid adjacent to the phase interface are compatible and coincide with the perturbations produced by the corrugations of the front, *i.e.*

$$\vec{\mathbf{U}}(t,\ 0^+,\ y) \equiv \sum_{j=1}^{3}(c_j\vec{\mathbf{L}}_j)e^{-i(\omega t+ky)} = \vec{\mathbf{V}}_2(t,\ y)\ . \tag{47}$$

Writing out (47) explicitly results in a system of three equations with unknowns being given by the coefficients c_j, and by the surface displacement Δ_0. Whether equations (47) are sufficient to determine the dispersion relation depends on the number of nonzero coefficients c_j or, equivalently, on the degree of "underdeterminacy" of a detonation front. This concept appears in a simple procedure which is used for discussing the stability properties of reaction fronts or discontinuity surfaces in general [2, 5]. The degree of "underdeterminacy" of a discontinuity can be calculated by counting the number of free parameters which could be associated with a small perturbation of the front (these are given by the number of sonic perturbations transmitted from the front[4], the entropy perturbation propagated in the downstream region of the flow and the surface displacement) minus the number of boundary conditions that the perturbation has to satisfy (in general there are three of these: conservation of mass, energy and momentum).

In the case of a strong or Chapman-Jouguet detonation there exist three free parameters (which correspond to the unknowns $c_1,\ c_2$ and to Δ_0) and the front is then with zero degree of "underdeterminacy". In this case, equation (47) has a solution provided that the determinant of the matrix of coefficients vanishes, *i.e.*

[4]In Figure 3 the number of sonic perturbations can be counted by using the characteristic curves for the different fronts summarized in Table I.

$$\det \begin{pmatrix} 0 & 1 & \delta p_2 \\ \\ 1 & -\dfrac{(\omega v_x + \tilde{l}_2)}{\gamma_2^2 w_2(\omega + \tilde{l}_2 v_x)} & \delta v_{x2} \\ \\ -\dfrac{\tilde{l}_1}{k} & -\dfrac{k}{\gamma_2^2 w_2(\omega + \tilde{l}_2 v_x)} & \delta v_{y2} \end{pmatrix} = 0 \ , \tag{48}$$

After some algebra, the general form of the dispersion relation is found to be

$$\begin{aligned} \frac{w_2}{\Gamma_- + \Gamma_+ v_2^2} \Bigg\{ i\frac{(v_1 - v_2)}{v_1} \Bigg[\omega^3 \frac{(1 - v_1 v_2)(\Gamma_+ - 2\gamma_2^2 v_2^2)}{v_2} + \omega^2 (1 - v_1 v_2)(\Gamma_+ - 2\gamma_2^2)\tilde{l}_2 \\ + \omega \left[2v_2(1 - v_1 v_2) - v_1 \left(\Gamma_- + \Gamma_+ v_2^2 \right) \right] \gamma_2^2 k^2 - \gamma_2^2 v_1 v_2 k^2 \left(\Gamma_- + \Gamma_+ v_2^2 \right) \tilde{l}_2 \Bigg] \\ + \sigma \frac{(\omega^2 - k^2)}{w_2} \left[\omega^2 \left(\Theta_2 - 1 - v_2^2 \right) - \frac{\omega}{v_2} \left(\Gamma_- + \Gamma_+ v_2^2 \right) \tilde{l}_2 + (1 + v_2^2) k^2 \right] \Bigg\} = 0 \ , \end{aligned} \tag{49}$$

which provides a relation $\omega = \omega(k)$ once the free variables v_1, v_2, c_{s2} and σ/w_2 are specified.

On the other hand, in the case of a weak detonation there exist four free parameters (corresponding to the unknowns c_1, c_2, c_3 and Δ_0) and this forces the introduction of a fourth boundary condition in order to make the solution evolutionary. In this respect, weak detonations are similar to weak deflagrations and in order to be fully determined require an equation describing the microscopic burning mechanism or, in the case of phase transitions, the rate of transformation of the old phase into the new one. This feature of weak detonations does not allow for a general discussion of their stability properties and restricts the analysis to the specific situations in which the fourth boundary condition can be expressed. For this reason, we will limit ourselves to discuss the general stability properties of strong and Chapman-Jouguet detonations only, remainding the study of weak detonations in cosmological phase transitions to a future work [32].

The next Sections will present the solution of (49) for the different cases of Chapman-Jouguet and strong detonations.

IV. Chapman-Jouguet and strong detonations

We here discuss the problem of potential corrugation instabilities behind the front of Chapman-Jouguet and strong detonations. Recalling the definition (26) of $\tilde{l}_2$, it is possible to see that a strong detonation naturally evolves into a Chapman-Jouguet detonation when the velocity behind the front passes from being subsonic to being equal to the sound speed.

We shall start by discussing this latter case, for which $\epsilon = 0$, $v_2 = c_s$ (hereafter $c_s \equiv c_{s2}$) and (49) reduces to

$$\frac{w_2}{\Gamma_- + \Gamma_+ c_s^2}\left\{ i\frac{(v_1 - c_s)}{v_1}\left[2\omega^3\frac{(1 - v_1 c_s)}{c_s} + 2\omega(1 - v_1 c_s)\gamma_2^2 c_s k^2\right]\right.$$
$$\left. +\sigma\frac{(\omega^2 - k^2)}{w_2}\left[\omega^2\frac{(1 + c_s^2)}{\gamma_2^2 c_s^2} + (1 + c_s^2)k^2\right]\right\} = 0\ . \tag{50}$$

Note that the dispersion relation (50) does not contain the eigenvalue $\tilde{l}_2$ (which is always multiplied by vanishing terms) and that, in order to avoid a singularity, the content of the curly brackets in (50) has to be zero. This condition can be imposed by requiring that

$$(\omega^2 + \gamma_2^2 c_s^2 k^2)\left[2i\omega\frac{(v_1 - c_s)(1 - v_1 c_s)}{v_1 c_s} + \sigma\frac{(\omega^2 - k^2)}{\gamma_2^2 c_s^2 w_2}\right] = 0\ , \tag{51}$$

which has the four distinct roots

$$\omega_{1,2} = \pm\, i\gamma_2 c_s k\ , \tag{52}$$

$$\omega_{3,4} = -\frac{\gamma_2^2 c_s w_2}{v_1 \sigma}\Bigg\{ i(v_1 - c_s)(1 - v_1 c_s) \mp \left[\left(\frac{v_1\sigma}{\gamma_2^2 c_s w_2}\right)^2 k^2 - (v_1 - c_s)^2(1 - v_1 c_s)^2\right]^{1/2}\Bigg\}. \tag{53}$$

We note that it is a common experience in dealing with the solution of dispersion relations that spurious roots can be introduced, which then need to be discarded on the basis of physical or conceptual considerations. An example of this is the root ω_1 which has positive imaginary part and would lead to an exponentially growing unstable mode. However, ω_1 should be rejected since it does not satisfy energy boundary conditions at short wavelengths and would lead to a "high frequency catastrophe". It is well known, in fact, that the surface energy associated with a perturbation of amplitude Δ is proportional to $\sigma k^2 \Delta^2$ and a cut-off wave number, above which instabilities are not allowed, is necessary in order to avoid accumulation of infinite energies at high frequencies.

The physical mechanisms which operate this limitation depend on the specific situation under examination and can be either dissipative effects, such as a fluid viscosity, or can be the consequence of surface tension. However, the root ω_1 does not contain any contribution coming from the surface tension and this has the consequence that even an infinitely stiff front (*i.e.* one with $\sigma \to \infty$) would appear to be unstable at all wavelengths. This behaviour suggests that the roots $\omega_{1,2}$ cannot provide a physical description of detonation fronts and will be discarded here. Further evidence of the inapplicability of the roots (52) comes from realizing that the term $(\omega^2 + \gamma_2^2 c_s^2 k^2)$ in (51) is the consequence of a Doppler frequency transformation relative to a medium moving at the sound speed. With a few simple calculations it is easy to see that this term should always be different from zero (see the Appendix).

After some algebraic manipulations (the details of which can be found in the Appendix), it is easy to show that the other two roots $\omega_{3,4}$, which are clearly dependent on σ, both have negative imaginary parts and therefore lead to *stable* solutions.

As discussed in Section III, this is further and more direct proof that at first order a Chapman-Jouguet detonation is *unconditionally stable*. We remark that this conclusion is in contrast with the results presented by Abney in [22] which indicated that Chapman-Jouguet detonations are effectively unstable at all wavelengths. As mentioned in the previous Section, the origin of this discrepancy is to be found in the approximations adopted in [22] for the derivation of the perturbed hydrodynamical quantities behind the detonation front, which have artificially removed the singular properties of this class of detonations.

The situation is not very different when the more general strong detonations are discussed. In this case, all of the expressions in the dispersion relation (49) need to be expanded around the sound speed, with terms up to the second order being retained. This is a consequence of the fact that at first order Chapman-Jouguet detonations and strong detonations are indistinguishable and a second order expansion is therefore necessary. The complete general dispersion relation, which results from lengthy algebraic manipulations, is presented in the Appendix. Here, we limit ourselves to discussing the equivalent expression obtained by setting $c_s^2 = 1/3$

$$\begin{aligned}\Bigg\{&\frac{2\sigma}{w_2}\left(\frac{4}{3}-7c_s\epsilon+4\epsilon^2\right)\omega^7+i\left[8c_s-v_1\left(1+\frac{1}{v_1^2}\right)(2+3c_s\epsilon)+12c_s\epsilon^2\right]\omega^6\\&-\frac{\sigma k^2}{w_2}\left(\frac{4}{3}-19c_s\epsilon+\frac{17}{2}\epsilon^2\right)\omega^5+ik^2\Bigg[4c_s-v_1\left(1+\frac{1}{v_1^2}\right)\\&+\left(10-\frac{3v_1c_s}{2}\left(5+\frac{7}{v_1^2}\right)\right)\epsilon+\left(54c_s-\frac{69}{4}\left(1+\frac{1}{v_1^2}\right)\right)\epsilon^2\Bigg]\omega^4\\&-\frac{\sigma k^4}{w_2}\left(\frac{4}{3}+5c_s\epsilon+\frac{\epsilon^2}{4}\right)\omega^3-i\frac{k^4\epsilon}{2}\left[1-3v_1c_s-\left(\frac{v_1}{4}\left(11+\frac{1}{v_1^2}\right)-\frac{13}{2}c_s\right)\epsilon\right]\omega^2\\&+\frac{3k^6\sigma\epsilon^2}{4w_2}\omega+i\frac{3k^6}{8}(c_s-v_1)\epsilon^2\Bigg\}\frac{w_2}{3\epsilon(2c_s+\epsilon)}=0\,.\end{aligned} \tag{54}$$

As for the case of (50), equation (54) can be satisfied only if the content of the curly brackets is set equal to zero and this results in a seventh order polynomial

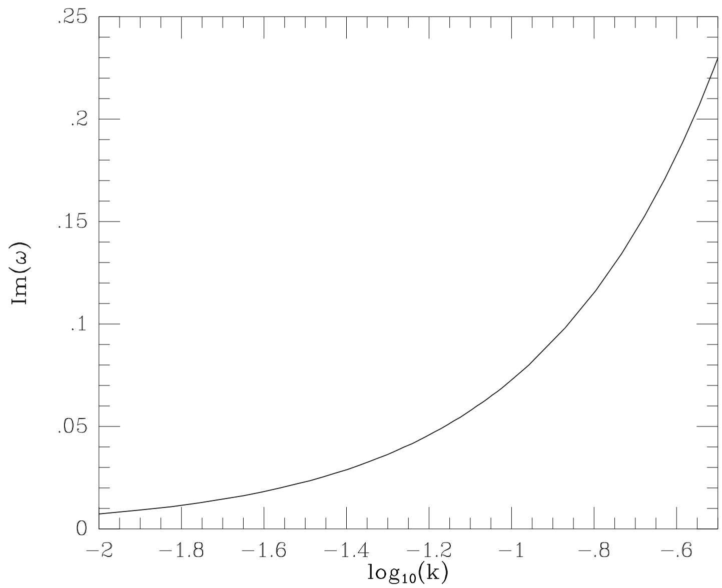
in ω with complex coefficients. (It is straightforward to check that (54) reduces to (51) when $\epsilon = 0$.) The roots of (54) can only be computed numerically and for this we have implemented an algorithm which makes use of a variant of Laguerre's Method (NAG Fortran Library C02AFF [33]). The computations can be performed only after all the parameters have been specified and for this purpose we have set $v_1 = (3/2)c_s$ and $\epsilon = -0.01$. We stress that only the solution of the complete polynomial allows one to deduce a consistent picture of the functional dependence of the growth rate on the wavenumber k. Any analysis which studies the dispersion relation (54) in the approximate expressions which it assumes in the long and short wavelength *limits* [22] can easily give rise to confusing outcomes.

In analogy with [22], we present in Figure 2 (a) results of the numerical computations for the long wavelength region and in Figure 2 (b) for the short wavelength one, with the wavenumber being expressed in units of w_2/σ. Figures 2 indicate that there is always a root of (54) which has positive imaginary part (all the others have either negative or constant imaginary part).

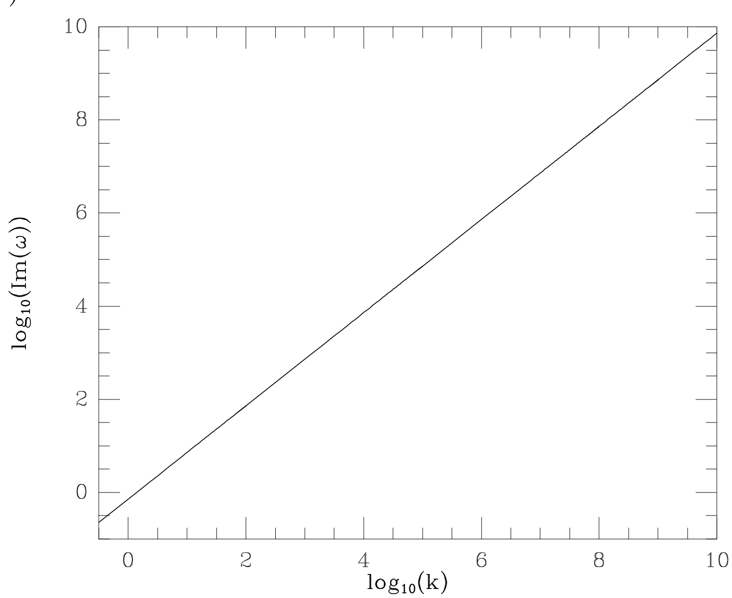
The instabilities described by this root have a growth rate which is independent of σ, increases linearly with the wavenumber k and produces an inevitable energy divergence at high frequencies. They clearly represent the relics of the unstable modes contained in the root ω_1 of the dispersion relation (51) and in analogy with what deduced for Chapman-Jouguet detonations, the results shown in Figures 2 refer to a physically inconsistent solution and should be discarded.

As a result, we are led to conclude that strong detonations are linearly *stable* to corrugations of the front and in this they resemble Chapman-Jouguet detonations to which they reduce for $\epsilon \to 0$. A similar result (*i.e.* that an evolutionary front is also stable with respect to corrugations), has been obtained also by Anile and Russo [34, 23, 26] in the context of the stability of shock waves. Proceeding within the theory of singular hypersurfaces, they came to the conclusion that the linear stability conditions for planar relativistic shock waves coincide with those obtained in the framework of corrugation stability.

(a)



(b)



Figures 2. Perturbation growth rate Im (ω) plotted as a function of the logarithm of the real wavenumber k expressed in units of w_2/σ. Figure 2 (a) refers to the long wavelength region, while Figure 2 (b) to the short wavelength one. The curves have been calculated assuming $v_1 = (3/2)\ c_s$, $c_s = 1/\sqrt{3}$ and $\epsilon = -\ 0.01$.

In the next Section we shall show that, in spite of these stability results, strong detonations cannot satisfy elementary boundary conditions and for this reason they should be ruled out from the scenario of cosmological first order phase transitions.

VI. Detonation fronts in cosmology

We here briefly investigate the consequences of the stability analysis performed in the previous Sections for a cosmological first order transition (as the electroweak or the quark-hadron) taking place by means of bubbles of the new phase growing as detonations moving into the old phase medium. It might be relevant to underline that the results of the stability analysis can find a consistent application only when the bubbles are sufficiently large so that they can be considered locally planar and this is the assumption we will follow hereafter.

Firstly, we consider the situation of a spherical bubble moving as a strong detonation; the results presented in Section IV indicate that these reaction surfaces are both evolutionary and stable with respect to corrugation instabilities. However, it is possible to demonstrate that such flow configurations cannot be realized during bubble growth as they cannot satisfy the required boundary conditions. Proofs of this have been given in [2] for nonrelativistic planar and spherical fronts, in [10] for relativistic spherical fronts and also in [6] for the case of relativistic planar fronts. The impossibility of having this class of reaction front can be shown also with more simple arguments. Consider a fluid element which is immediately behind a strong detonation and which has been just transformed into the new phase (we here make the implicit assumption that the sound speeds are constant on either side of the front). This fluid element, which was initially at rest, has been put into motion by the front which will be seen by the fluid element as receding at a subsonic velocity. Symmetry and the presence of an origin for the bubble, require that the fluid velocity behind the front becomes zero somewhere in the flow profile and this could occur either via a rarefaction wave, or via a shock wave. However, the front edge of the

rarefaction wave or the discontinuity surface would be seen as moving at the sound speed or faster relative the fluid element and as a consequence either one would inevitably overtake the detonation front. As a result, neither of these two flows can be established behind a strong detonation front which is then unable to adjust itself to the required boundary conditions and so cannot be produced in practice.

As mentioned in Section IV, Chapman-Jouguet detonations represent a limiting case of the more general strong detonations and share with the latter the properties of stability with respect to corrugations of the front. In the case of Chapman-Jouguet detonations however, the boundary conditions for the medium behind can be suitably satisfied (the front is moving at the sound speed as seen from the medium behind) and it is easy to show that the detonation front is then followed by a rarefaction wave which progressively decelerates and decompresses the fluid in the new phase [2, 1, 10]. The occurrence of cosmological Chapman-Jouguet detonations is however made difficult because of the considerable amount of supercooling required before the nucleation of the new phase bubbles and because they would tend to leave the new phase in a superheated state [12, 16, 18].

Next, we consider the case of a phase interface in a cosmological phase transition moving as a weak detonation. This scenario has recently been re-examined by Laine [6, 35] who has proposed the possibility of considering such fronts as a "natural mechanism for bubble growth in phase transitions" in view of the differences between chemical burning (in which context they are excluded) and cosmological phase transitions.

Boundary conditions for a weak detonation can be easily satisfied since the flow behind the front can be suitably "adjusted" by means of a rarefaction wave which progressively slows down the new phase, bringing it to rest at the rear edge of the wave which behaves as a weak discontinuity moving at the local sound speed. Moreover, weak detonations require a degree of supercooling smaller than the one needed by Chapman-Jouguet detonations and would also produce a smaller or zero superheating of the low temperature phase. As discussed in Sections II and III, a general discussion of the stability properties of weak detonations is not possible

because of the intrinsic “underdeterminacy” of these fronts. However, we want to recall the attention on a general a feature of the causal structure of weak detonations that should be taken into account when performing a stability analysis.

Generally speaking, it is not implausible to expect that higher order effects could intervene during the growth of linear instabilities so as to modify the energy balance and saturate the oscillations which could then be present but would not grow unboundedly with time. A similar argument is rather appropriate for many instability phenomena and has been used also in relation to cosmological deflagrations [14] and Chapman-Jouguet detonations [22]. It is essential to stress however, that a non-linear saturation produced by mutual interaction between the front and the adjacent fluid cannot be invoked in the case of weak detonation fronts, because of the intrinsic causal structure of a weak detonation.

In order to clarify the arguments, in Figure 3 we have shown schematically the causal structure of the six types of reaction front which were summarized in Table I. The figure consists of six different spacetime diagrams, (all referred to the front rest frame), with time on the vertical axis and the space coordinate on the horizontal one. For each diagram, the thick solid line represents the worldline of a fluid element passing from region 1 to region 2, the thin solid and arrowed lines denote the characteristic curves relative to the front[5] [$\mathcal{C}^j_\pm$ are the forward and the backward characteristics of the regions ahead of the front ($j = 1$) and behind it ($j = 2$)], the dotted line shows the the sound speed in the frame of the front and the dashed line represents the worldline of the front. The characteristic curves are drawn so to reach the front when the fluid element crosses it and to depart from the front at the same instant; note that for simplicity we have assumed the sound speeds to be the same on both sides of the front.

[5]We recall that the characteristic curves can be interpreted as the directions in spacetime along which sonic perturbations are transmitted. In this sense it is possible to define the “region of determinacy” of an event $\mathcal{P}$ as the region of spacetime included between the characteristic curves converging to the point $\mathcal{P}$.

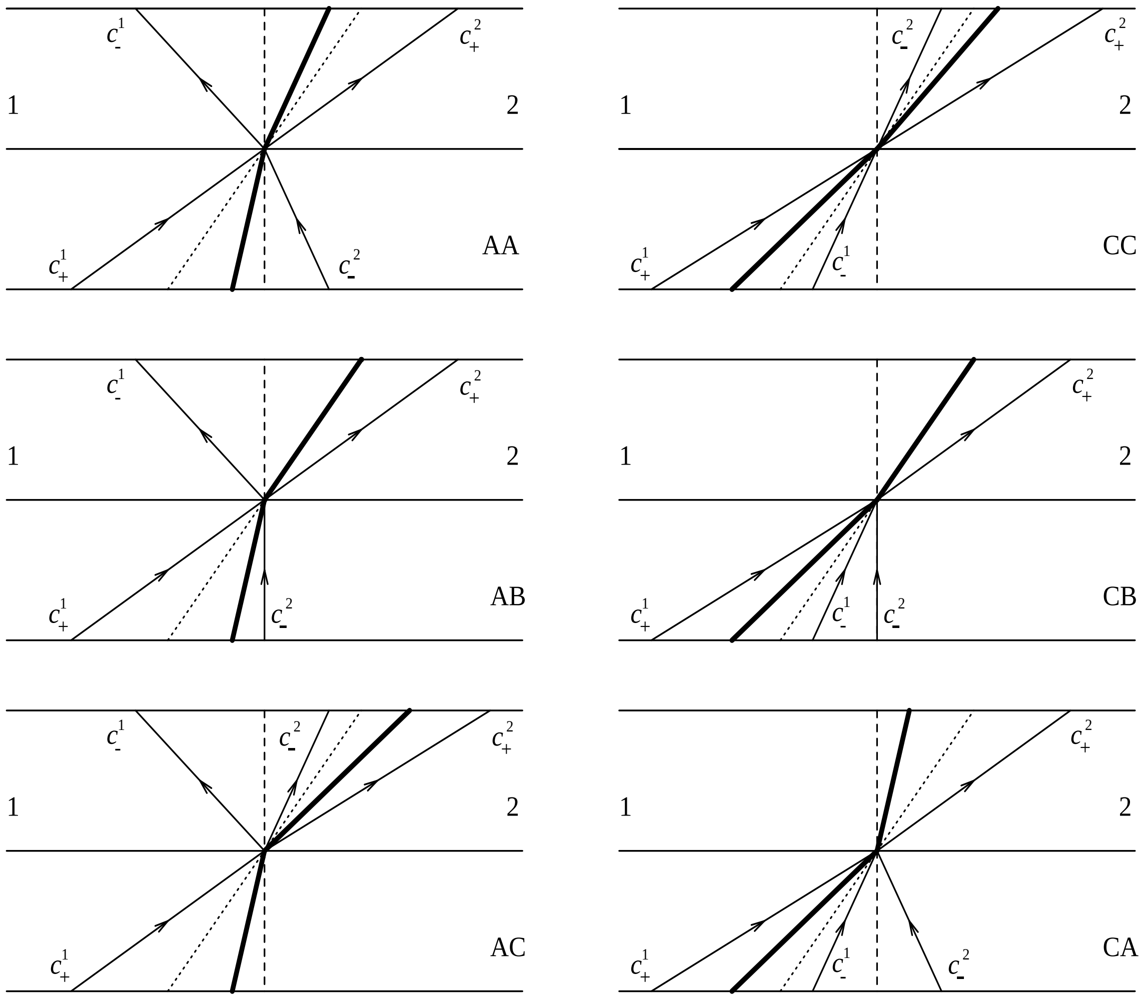


Figure 3. Spacetime diagrams and characteristic representations of the various types of reaction front (drawn as left propagating). Deflagrations are represented on the left and detonations on the right; the six different diagrams refer respectively to: AA: weak deflagration, AB: Chapman-Jouguet deflagration, AC: strong deflagration, CC: weak detonation, CB: Chapman-Jouguet detonation and CA: strong detonation. The diagrams are drawn relative to the front rest frame with time on the vertical axis and the space coordinate on the horizontal one. The thick solid line represents the worldline of a fluid element passing from region 1 to region 2. For each diagram, the characteristic curves are denoted with thin solid and arrowed lines ($\mathcal{C}^j_{\pm}$, with $j = 1,\ 2$, are the forward and the backward characteristics relative to the regions ahead of and behind the front respectively), the sound speed in the frame of the front is marked with a dotted line, while the dashed line represents the worldline of the front. (For simplicity we have assumed the sound speeds to be the same on both sides of the front.)

Deflagrations are represented in the left column, detonations in the right one and the different diagrams refer respectively to AA: weak deflagration, AB: Chapman-Jouguet deflagration, AC: strong deflagration, CC: weak detonation, CB: Chapman-Jouguet detonation and CA: strong detonation. (The letters A, B, C, express the fact that the fluid element can move, relative to the front, at a speed smaller, equal or larger than the sound speed, with the first letter referring to the medium ahead and the second one to the medium behind.)

With this representation, the diagrams in Figure 3 show that there is always a mutual causal connection between the front and the medium ahead of it when the latter is subsonic (as in the diagrams AA, AB and AC) and the worldline of the fluid element is within the region of determinacy of the front. In this case the front can transmit information upstream in region 1 by means of the backward characteristic $\mathcal{C}_-^1$. This mutual causal connection is not present when the medium ahead is supersonic (as in the diagrams CC, CB and CA) in which case the front cannot influence the incoming flow and both the characteristic curves $\mathcal{C}_+^1$ and $\mathcal{C}_-^1$ are directed towards the front. This latter property shows that corrugation instabilities of detonations fronts cannot propagate in the upstream region 1.

Similarly, from Figure 3 it is possible to deduce the causal connection between the front and the medium behind it, by looking at whether there is any backward characteristic curve $\mathcal{C}_-^2$ reaching the front from region 2. It is evident that this can occur only if the downstream flow is subsonic (as in diagrams AA and CA) or sonic (as diagrams AB and CB); this latter case represents a limiting situation, with the mutual causal connection being just marginal as shown by the characteristic $\mathcal{C}_-^2$ being tangent to the worldline of the front. The front has no mutual causal connection with the medium behind when the downstream flow is supersonic (as in the diagrams AC and CC) and in this case the characteristic curves $\mathcal{C}_+^2$ and $\mathcal{C}_-^2$ are both in the direction of the flow. This difference is a fundamental one, establishing that there is a mutual causal connection between the front and the medium behind only when the flow out of it is subsonic or sonic. As a consequence, for subsonic and sonic downstream flows, the medium behind *can* respond to any perturbation

produced by the front and, at least in principle, it can counteract the growth of the potential instabilities. This backreaction could then appear in the nonlinear effects mentioned earlier and might be at the origin of the formation of the typical corrugated but stable cellular flames observed in laboratory experiments [36, 3], which are produced by weak deflagrations.

This stabilizing mechanism *cannot* operate in weak detonations, since in this case there is no mutual interaction between the front and the media either ahead of it or behind it. We remark that this argument is simply based on the causal structure of weak detonations and is therefore independent of the order at which the perturbation analysis is performed.

In the context of causal connection, Chapman-Jouguet detonations represent a limiting case but do not suffer from the causality problems discussed for weak detonations. As shown in the diagram CB of Figure 3 in fact, the front is just marginally mutually connected with the medium behind it and this could then provide the back-reaction required for saturating the potential perturbations produced at front.

VII. Conclusion

We have performed a linear stability analysis of relativistic strong and Chapman-Jouguet detonations in a relativistic fluid with specific attention being paid to the stability properties of these fronts with respect to corrugations. This work has been stimulated by a recent stability analysis of relativistic Chapman-Jouguet detonations presented by Abney [22] with reference to the mechanisms of bubble growth in first order cosmological phase transitions. In order to re-examine the above analysis and extend it to strong detonation fronts, we have implemented the standard linear perturbation techniques on the hydrodynamical equations for the fluids on either side of the front. The new perturbed expressions have then been required to be compatible with the conservation of energy and momentum across a

front subject to a corrugation perturbation. By doing this, we recovered the result that while no perturbations can be present in the fluid ahead of the front, the growth of instabilities cannot be ruled out for the medium behind.

The study of the time evolution of these potential has been complicated by the presence of a singular behaviour of the perturbed hydrodynamical variables if the fluid velocity out of the front is equal to the local sound speed. This feature, which is common in corrugation stability studies [2, 23] and which did not emerge in the previous work of Abney [22], has been circumvented by using an expansion of the dispersion relations around the sonic point.

As a result of the analysis, we have found that strong detonations can be both evolutionary and stable with respect to corrugations of the front, while Chapman-Jouguet detonations appear to be unconditionally linearly stable fronts. Our conclusions are in contrast with the ones presented by Abney in [22] which indicated that Chapman-Jouguet detonations are effectively unstable at all wavelengths. We believe that the approximations adopted in [22] for the derivation of the perturbed hydrodynamical quantities behind the detonation front are at the origin of this difference. Because of their intrisic "underdeterminacy", generic weak detonations cannot be studied, but a further equation accounting for the entropy jump across the front needs to be provided. This feature unfortunately restricts the stability analysis to specific situations for which a fourth boundary condition can be expressed and this will be the subject of a future investigation [32].

The results of the stability analysis have been related with the problem of boundary conditions which emerges when detonation fronts are used in the scenario of cosmological first order phase transitions such as the electroweak and quark-hadron transitions. In this context, strong detonations cannot satisfy the required boundary conditions, despite their stability properties, and therefore are not suitable for describing the growth of bubbles of the new phase. Conversely, Chapman-Jouguet detonations do not suffer restrictions being imposed by the boundary conditions or by the growth of corrugation instabilities but require a considerable amount of supercooling of the new temperature phase. Weak detonations, on the other hand,

can satisfy boundary conditions for the growth of spherical bubbles and could be induced with rather small amounts of supercooling. However, their causal structure is such that the lack of a *mutual connection* between the front and the media on either side, would not allow for higher order effects to intervene so as to stabilize the instabilities that could have possibly been produced. This feature of weak detonations represents a strong motivation for the study of their stability properties and probably hints to the inadequacy of laminar flow approaches for the study of these fronts.

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Appendix

In this Appendix, details are presented of different expressions which are discussed in the main text of the paper. The first one concerns the stability properties of Chapman-Jouguet detonations as they are deduced from the solutions (53) of the dispersion relation (51). The second one is focussed on the presentation of the complete and generic expressions of the dispersion relation (49) in the case of strong detonations.

Let us start with considering the form of the roots $\omega_{3,4}$ of (51) which can also be written as

$$\omega_{3,4} = -k_c \left[i \mp \left(\frac{k^2}{k_c^2} - 1 \right)^{1/2} \right] , \tag{55}$$

where k_c is a critical wavenumber defined as

$$k_c = \frac{\gamma_2^2 c_s w_2 (v_1 - c_s)(1 - v_1 c_s)}{v_1 \sigma} . \tag{56}$$

Since k_c is a positive real number, the sign of the square root in (55) depends on whether the wavenumber for the perturbation mode k is larger or smaller than the critical wavenumber. The imaginary part of $\omega_{3,4}$ can then be

$$\mathrm{Im}\ \omega_{3,4} = -k_c < 0 , \qquad \text{if } k \geq k_c , \tag{57}$$

or

$$\mathrm{Im}\ \omega_{3,4} = -k_c \left[1 \mp \left(1 - \frac{k^2}{k_c^2} \right)^{1/2} \right] , \qquad \text{if } k < k_c . \tag{58}$$

It is easy to see that in both cases and irrespective of which of the two roots is chosen, the imaginary part of the solutions of the dispersion relation is always negative, thus establishing the stability properties of of Chapman-Jouguet detonation.

Next we turn to the general expression of the dispersion relations which are deduced after lengthy but straightforward algebraic manipulations. In the case of strong detonation fronts and writing $v_2 = c_s + \epsilon$, (with ϵ being suitably small), equation (49) becomes

$$\Bigg\{ i\left(1-\frac{c_s+\epsilon}{v_1}\right)\Bigg\{\left[(1-v_1c_s)\frac{2}{c_s}-\frac{(1+v_1c_s)}{c_s^2}\epsilon+\frac{(c_s^4-2c_s^2+1)}{c_s^3(1-c_s^2)^2}\epsilon^2\right]\omega^3$$
$$+k^2\Bigg\{\frac{2c_s(1-v_1c_s)}{1-c_s^2}+\frac{[c_s(3+c_s^2)+v_1(2c_s^4-7c_s^2+1)]}{c_s(1-c_s^2)^2}\epsilon$$
$$+\frac{[c_s(c_s^4+14c_s^2+1)+v_1(c_s^6-26c_s^4-7c_s^2-4)]}{2c_s^2(1-c_s^2)^3}\epsilon^2\Bigg\}\omega$$
$$+\left[\frac{v_1c_s}{1-c_s^2}\epsilon+\frac{[v_1(3+2c_s^2)-c_s]}{2(1-c_s^2)^2}\epsilon^2\right]\frac{k^4}{\omega}-\frac{v_1k^6c_s^2}{2(1-c_s^2)^2\omega^3}\epsilon^2\Bigg\}$$
$$+\frac{\sigma}{w_2}\Bigg\{\left[(1-c_s^4)c_s^2-c_s(2c_s^4+c_s^2+1)\epsilon+(1-c_s^4)\epsilon^2\right]\frac{\omega^2}{c_s^4}$$
$$+(c_s^2+1)k^2+\frac{(2c_s^2+1)k^2}{c_s}\epsilon+\frac{(2c_s^2-1)k^2}{2c_s^2}\epsilon^2-\frac{k^4}{2(1-c_s^2)\omega^2}\epsilon^2\Bigg\}\Bigg\}\frac{1}{\epsilon(2c_s+\epsilon)}=0\,, \tag{59}$$

which reduces to (54) when $c_s^2 = 1/3$.

At last, we show that the quantity $(\omega^2+\gamma_2^2c_s^2k^2)$ in equation (51) should always be different from zero (a similar statement can be found in [15]). Because in the Newtonian case the expressions are simpler and can be handled analytically we will give the proof in this limit of small velocities. However, the extension of the result to the special relativistic case is rather straightforward. As discussed in Section III, the dispersion relation can be obtained once requiring that the perturbed state vector satisfies the junction conditions for the energy and momentum. In the limit of small velocities, we can neglect all the γ Lorentz factors and the components of the perturbed state vector can be simply written as [we here omit a common factor $\exp\,[-i(\omega t+k_y)]$ and drop the indices referring to region 2]

$$
\begin{aligned}
\delta p &= c_2\gamma^2 w\frac{(\omega+lv_x)}{(\omega v_x+l)}e^{-ilx} \simeq c_2 w\frac{(\omega+lv_x)}{l}e^{-ilx}\\
\delta v_x &= c_1 e^{i(\omega/v_x)x}+c_2 e^{-ilx}\\
\delta v_y &= c_1\frac{\omega}{v_x k}e^{i(\omega/v_x)x}+c_2\frac{k}{(\omega v_x+l)}e^{-ilx} \simeq c_1\frac{\omega}{v_x k}e^{i(\omega/v_x)x}+c_2\frac{k}{l}e^{-ilx}
\end{aligned}
\tag{60}
$$

where $l \equiv \tilde{l}_2$ and we have used $\tilde{l}_1 = -\omega/v_x$. Note that we have here exploited the possibility for a different normalization of the eigenvectors $\vec{\mathbf{L}}_1,\ \vec{\mathbf{L}}_2$ and we are not restricting the discussion to the case $x = 0$.

Imposing the condition (47) and asking for the determinant of the coefficients to be zero is equivalent to set

$$
\det\begin{pmatrix}
0 & w\dfrac{(\omega+lv_x)}{l}e^{-ilx} & \delta p\\
e^{i(\omega/v_x)x} & e^{-ilx} & \delta v_x\\
\dfrac{\omega}{v_x k}e^{i(\omega/v_x)x} & \dfrac{k}{l}e^{-ilx} & \delta v_y
\end{pmatrix} = 0\ . \tag{61}
$$

A solution of (61) is clearly given by

$$
\omega = -lv_x = iv_x k\ , \tag{62}
$$

where we have used, from (22), that $l = l_2 \simeq -ik$. However, the solution (62), which represents the small velocity limit of the positive root of (52), is a spurious one and should discarded since it would imply that $c_1+c_2 = 0$, and the corresponding solution (60) would be then identically zero.

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